An Extension of Multiple Cosmic String Solution: A Proposal

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Abstract We extend the work done for cosmic strings on the perturbative calculation of vacuum polarization of a massless field in the space-time of multiple cosmic strings and show that for a more general class of locally flat metrics the one loop calculation do not introduce any new divergences to the VEV of the energy of a scalar particle or a spinor particle. We explicitly perform the calculation for the configuration where we have one cosmic string in the presence of a dipole made out of cosmic strings both for the scalar and the spinor cases.

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1 Introduction

Extensive work has been done on cosmic strings /1 and other topological defects. A remarkable property of the field theoretical calculations using cosmic strings as background is the existence of a finite contribution to the energy of a scalar particle due to the presence of the cosmic string /2. In a beautiful paper /3 A.N. Aliev showed that there is an intricate cancellation mechanism in the cosmic string background which makes the perturbative expression finite. The divergence due to the one loop integration is cancelled by the zero coming from the anti Fourier transforming the resulting expression.

In this note we want to point out to a simple property of certain metrics, made out of two "holomorphic" structures, where similar cancellation occurs. In this sense we extend the Aliev result to more general, still locally flat, metrics. These metrics also have a special property. If the euclidean Dirac operator is written in the background of such a metric, it is the true square root of the Klein-Gordon operator, a property which the free Dirac operator has in flat space. If we take a special form of these metrics , this property also holds for the Dirac operator with the lorentzian signature. Thus our metrics generalize an important property of the free Dirac operator which exists in flat space metric. Our metrics are all locally flat, with possible Dirac delta function type singularities.

We introduce our new set of metrics in Section II and analyze their properties. Section III is devoted to the calculation of the vacuum fluctuations of a massless scalar particle in the background of a representative member of our metrics. In Section IV , we perform the similar calculation for a spinor particle. We conclude with a few remarks.

2 New metrics

Take the special form of the Dirac operator

$$D' = e^{-g_1(x_0-\gamma^0\gamma^3x_3)}(\gamma^0\partial_0+\gamma^3\partial_3)+e^{-g_2(x_1-\gamma^1\gamma^2x_2)}(\gamma^1\partial_1+\gamma^2\partial_2). \tag{1}$$

Here γ^μ are Dirac gamma matrices obeying the anticommutation relations $[\gamma^\mu,\gamma^\nu]_+=2g^{\mu\nu}$, and g_1, g_2 are arbitrary smooth functions which are defined by their power expansions. Define the d'Alembertian operator as

$$\Box = \frac{TrD'D'}{4}. \tag{2}$$

One sees that the d'Alembertian operator reads

$$\Box = e^{-f_1(x_0,x_3)}\left(\frac{\partial^2}{\partial x_0^2}-\frac{\partial^2}{\partial x_3^2}\right)-e^{-f_2(x_1,x_2)}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right), \tag{3}$$

where

$$f_1(x_0,x_1) = \frac{Tr}{4}\Big(g_1(x_0+\gamma^0\gamma^3x_3)+g_1(x_0-\gamma^0\gamma^3x_3)\Big) \tag{4}$$

$$f_2(x_1,x_2) = \frac{Tr}{4}\Big(g_2(x_1+\gamma^1\gamma^2x_2)+g_2(x_1-\gamma^1\gamma^2x_2)\Big). \tag{5}$$

One can derive the metric that will give this d'Alembertian operator, using the relation $\Box = \frac{1}{\sqrt{-g}}\partial_\mu g^{\mu\nu}\sqrt{-g}\partial_\nu$. This d'Alembertian can be derived from the metric

$$ds^2 = e^{-f_1(x_0,x_3)}(dx_0^2-dx_3^2)-e^{-f_2(x_1,x_2)}(dx_1^2+dx_2^2) \tag{6}$$

where f_1, f_2 are defined as in (4, 5).

Note that the metrics given in equation (6) are all locally flat, vacuum solutions, allowing possible Dirac delta function type singularities. The metric is locally flat , since we can define new coordinates,

$$d\tau = e^{g_1(x_0+x_3)}\frac{(dx_0+dx_3)}{\sqrt{2}} \tag{7}$$

$$d\overline{\tau} = e^{g_1(x_0-x_3)}\frac{(dx_0 - dx_3)}{\sqrt{2}} \tag{8}$$

$$d\zeta = e^{g_2(x_1+ix_2)}\frac{(dx_1 + idx_2)}{\sqrt{2}} \tag{9}$$

$$d\overline{\zeta} = e^{g_2(x_1-ix_2)}\frac{(dx_1 - idx_2)}{\sqrt{2}}. \tag{10}$$

Then

$$ds^2 = d\tau d\overline{\tau} - d\zeta d\overline{\zeta}, \tag{11}$$

a metric which is locally flat. Only zeros and singularities of g_1 and g_2 can introduce curvature at certain points, giving rise to Dirac delta function type singularities. For particular choices of g_1 and g_2, we see that to perform the transformation from ζ and τ back to x_1+ix_0 and x_0+x_3, we have to introduce cuts to the τ and ζ planes.

For a given metric with of the form given in (6), with a general function f_1 and f_2 the Dirac operator can be derived using standard procedure.

$$\begin{aligned} D = {} & e^{\frac{f_1(x_0,x_3)}{2}}[\gamma^0\partial_0 + \gamma^3\partial_3 - \frac{\gamma^0}{4}\partial_0 f_1 + \frac{\gamma^3}{4}\partial_3 f_1] \\ & + e^{\frac{f_2(x_1,x_2)}{2}}[\gamma_1\partial^1 + \gamma^2\partial^2 + \frac{\gamma^1}{4}\partial_1 f_2 + \frac{\gamma^2}{4}\partial_2 f_2]. \end{aligned} \tag{12}$$

For this general case, the square of the Dirac operator does not give us the d'Alembertian. There are extra terms proportional to derivatives of f_1 and f_2. We state this fact by writing

$$\Box \neq \frac{Tr}{4}DD. \tag{13}$$

If we choose $f_1 = 0$ and f_2 so that it can be written as in eq.(5), we get

$$\frac{\gamma^1}{4}\frac{\partial g_2}{\partial x_1} + \frac{\gamma^2}{4}\frac{\partial g_2}{\partial x_2} \equiv 0. \tag{14}$$

Then $D' = D$ for $f_1 = 0$ and eq. (6) is satisfied with D as well for this special case. To derive this result we used the series expansion for g_2 and the

relation $\gamma^1\gamma^2\gamma^1\gamma^2 = -1$. Note that if we take a metric with the Euclidean , instead of the Lorentzian signature, this behaviour can also be generalized to the case where f_1 is an arbitrary function, in the form given as in eq. (4). With this choice for f_1 and f_2 the Dirac operator is the true square root of the d'Alembertian, similar to its behaviour in the free case. We will not treat this case in our work, though, and take $f_1 = 0$.

Our examples are extensions of the free case, since our metrics are all locally flat. One should recall that solutions with only topological defects also have this property. This gives us the possibility of finding interesting solutions which generalize the cosmic string solutions and calculating vacuum fluctuations in their background. Below, using semi-classical methods, we will study two cases where the scalar and the spinor field is coupled to the metric. We explicitly study the simplest case which corresponds to a 'dipole'. The other cases can be treated along similar lines.

Note that if we do not want to extend our calculations to spinors, we can replace $\gamma^1\gamma^2$ by $i = \sqrt{-1}$ and $\gamma^0\gamma^3$ by unity, where both are multiplied by the unit matrix.

3 Scalar field

A.N.Aliev [/3] has shown that at first order perturbation theory, the cosmic string gives a finite contribution to the vacuum energy. This result is in accordance with the expectations, since it is well known that even the exact result does not introduce additional infinities. Our result is the extension of the Aliev result. The multiple cosmic string solution is a special case of our case since we can write

$$\sum_{i=1}^{n} \log(x_{1i}^2 + x_{2i}^2) = \frac{Tr}{4} \left(\sum_{i=1}^{n} (\log(x_{i1} + \gamma^1\gamma^2 x_{i2}) + \log(x_{i1} - \gamma^1\gamma^2 x_{i2})) \right), \quad (15)$$

where the LHS of the equation corresponds to the multiple-cosmic string solution [/4]. The vacuum fluctuations for the multiple cosmic string are cal-

culated in references 3 and 5.

We can show that the similar finite result in first order perturbation theory can be obtained if f_1, f_2 are taken in the form given in eq.s(4,5). Then the d'Alembertian operator is of the form given in equation(3).

To illustrate how this mechanism works, we will use this formalism to calculate the vacuum fluctuations for one special form, which can be interpreted as cosmic strings which result in angle defects and excesses. The same cancellation mechanism works for the other cases as well.

In our example we take $f_1 = 0$

$$g_2 = -\beta \log\left(1 + \alpha(z - \frac{1}{z})\right) \tag{16}$$

When we study only the scalar particle case, we can take $z = x_1 + ix_2$. If we want to extend our problem also to the spinor case, we replace z by $\zeta = x_1 + \gamma^1\gamma^2 x_2$. and use the $Trace$ operation at appropriate points. We write

$$\Box = \partial_0^2 - \partial_3^2 - 4e^{\beta\log(1+\alpha(z-\frac{1}{z}))+\beta\log(1+\alpha(\overline{z}-\frac{1}{\overline{z}}))}\frac{\partial}{\partial\overline{z}}\frac{\partial}{\partial z} \tag{17}$$

In first order perturbation theory we first expand the exponential and then the logarithm. We end up with

$$\Box_1 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 - \beta\alpha(2x_1 - \frac{2x_1}{x_1^2 + x_2^2})(\partial_1^2 + \partial_2^2). \tag{18}$$

The first order Greens Function reads

$$G_F^{(1)}(x-y) = \int dw G_F^{(0)}(x-w)V(w)G_F^{(0)}(w-y) \tag{19}$$

where

$$V = V_1(x_1, x_2)(\partial_1^2 + \partial_2^2) \tag{20}$$

and

$$V_1(x_1, x_2) = -\beta\alpha(2x_1 - \frac{2x_1}{x_1^2 + x_2^2}). \tag{21}$$

If we go to momentum space we have to calculate

$$\int dq \int dp \frac{p_1^2 + p_2^2}{p^2(p-q)^2} e^{iqx} V(q) \tag{22}$$

to obtain the $G_F^{(1)}$ at the coincidence limit .To get $< T_{00} >$, the VEV of the energy density, we have to differentiate $G_F^{(1)}$. This operation can be shown to result in the integral

$$\int dq \int dp \frac{(p_1^2 + p_2^2)^2}{p^2(p-q)^2} e^{iqx} V(q) \tag{23}$$

where

$$V(q) = \delta(q_0)\delta(q_3) \int dx_1 dx_2 V_1(x_1, x_2) e^{i(q_1 x_1 + q_2 x_2)} \tag{24}$$

$V_1(x_1, x_2)$ was defined in eq. (21).

As Aliev has shown /3 this calculation boils down to multiplying $V_1(q)$ by $(q_1^2 + q_2^2)^2$ and anti Fourier transforming the result. This gives us

$$< T^{00} >= -2A_1 \frac{\alpha\beta x_1}{(x_1^2 + x_2^2)^3} \tag{25}$$

for x_1 very much smaller than unity, A_1 a finite constant, α, β as in eq.(21).

There are no infinities in this order, since the divergence of the integral

$$\int d^4p \frac{(p_1^2 + p_2^2)}{p^2(p-q)^2} \tag{26}$$

has been cancelled exactly by the zero of the anti Fourier transforming the result. Here we have taken formulae like

$$\int d^2p e^{ip_1x_1 + ip_2x_2}(p_1^2 + p_2^2) = C \frac{\epsilon}{(x_1^2 + x_2^2)^2} \tag{27}$$

$$\int d^2p e^{ip_1x_1 + ip_2x_2} \log(p_1^2 + p_2^2) = C'(x_1^2 + x_2^2)^{-1}(1 + C_1 \epsilon \log(x_1^2 + x_2^2)) \tag{28}$$

as given in /6, and used dimensional regularization. Here ϵ is the parameter that goes to zero in the dimensional regularization and C, C', C_1 are three finite constants.

Our example, with the choice for g_2 given by equation (16) corresponds to three cosmic strings. Two cosmic strings which give rise to defect angles located at x_1 and x_2 and one cosmic string which which results in an excess angle, located at the origin. Here x_1 and x_2 are given as

$$x_1 = -\frac{1}{2\alpha} - \frac{1}{2\alpha}\sqrt{1+4\alpha^2}, \qquad x_2 = 0, \tag{29}$$

$$x_1 = -\frac{1}{2\alpha} + \frac{1}{2\alpha}\sqrt{1+4\alpha^2}, x_2 = 0. \tag{30}$$

We have in total one cosmic string and a "dipole" made out of cosmic strings.

Although we have explicitly treated one specific example, one can show that the divergence cancellation occurs in the other cases where equations f_1 and f_2 are satisfied as well. We have to fourier transform V_1 first. This operation may produce ϵ if V_1 is a monomial in x^2. If we have negative powers of x^2, we may not get an ϵ; however, then we get positive powers of q^2. Upon anti fourier transforming this expression, we get the necessary ϵ factor multiplying our expressions. We have exprienced that, if we do not take f_1 and f_2 as given in equations (4) and (5), we do not get the necessary power of ϵ to cancel the one coming from the p integration. The metric is not necessarily locally flat then, though, and we do not anticipate his cancellation in the first place.

4 Spin 1/2 case

We can calculate the vacuum fluctuation of a spinor in the presence of the same configuration. We find that the energy of the spinor particle is of the same form as the scalar case.

The vacuum expectation value of energy-momentum tensor for spin 1/2 case is given by/[7]

$$< T_{\mu\nu}(s = \frac{1}{2}) >= \frac{i}{2} < [\overline{\psi}(x)\gamma_{(\mu}\nabla_{\nu)}\psi(x') - (\nabla_{(\mu}\overline{\psi}(x')\gamma_{(\nu)}\psi(x')] > \tag{31}$$

for Dirac spinors. The Feynman propagator S_F is given as a time ordered product

$$S_F(x,x') = \; i < 0|T(\overline{\psi}(x')\psi(x))|0 > . \tag{32}$$

For the vacuum expectation value of the energy momentum tensor we have the relation

$$< T_{00}(x) >= -Tr[\partial_0 S_F]. \tag{33}$$

Feynman Green function $S_F(x,x')$ satisfies the relation

$$i\gamma^\mu(\partial_\mu - \Gamma^\mu)S_F(x,x') = \; \frac{1}{\sqrt{-g}}\delta^4(x-x') \tag{34}$$

where Γ_μ are spin connection terms for the curved space-time. In eq.(6) the Dirac operator has the form

$$D = \gamma^0\partial_0 + \gamma^3\partial_3 + e^{f_2/2}[\gamma^1\partial_1 + \gamma^2\partial_2 - \frac{1}{4}(\gamma^1\partial_1 - \gamma^2\partial_2)] \tag{35}$$

and if we take f_2 as a function of $f_2(x_1 - \gamma^1\gamma^2 x_2)$ last term in equation(35) vanishes and square of the Dirac equation gives the d'Alembertian of the spin-0 case:

$$D^2 = \partial_0^2 - \partial_3^2 - e^{\frac{(f_2+\overline{f}_2)}{2}}(\partial_1^2 + \partial_2^2) \tag{36}$$

Now spin-1/2 Green function S_F obeys the equation

$$i[\gamma^0\partial_0 + \gamma^3\partial_3 + e^{f_2/2}(\gamma^1\partial_1 + \gamma^2\partial_2)]S_F = \frac{1}{e^{-f_2}}\delta^4(x,x') \tag{37}$$

This expression can be written in the form of

$$\begin{aligned} i\gamma^\mu\partial_\mu S_F = \delta^4(x,x') + i[(1-e^{-f_2})(\gamma^0\partial_0 + \gamma^3\partial_3) \\ + (1-e^{-f_2/2})(\gamma^1\partial_1 + \gamma^2\partial_2)]S_F \end{aligned} \tag{38}$$

which is more convenient for the perturbative approach. In the perturbation series, the solution of the equation (38) can be given as

$$S_F(x,x') = S_F^{(0)}(x,x') + S_F^{(0)}VS_F^{(0)} + \cdots \tag{39}$$

with

$$V = if_2(\gamma^0\partial_0 + \gamma^3\partial_3 + \frac{1}{2}(\gamma^1\partial_1 + \gamma^2\partial_2)). \tag{40}$$

$S_F^{(0)}$ is the flat space-time spin 1/2 Green function and has the form

$$S_F^{(0)}(x,x') = \int \frac{d^4k}{(2\pi)^4}\frac{\gamma k}{k^2}e^{-ik(x-x')}. \tag{41}$$

First order correction to the Green function can be evaluated from $S_F^{(0)}$

$$S_F^{(1)} = S_F^{(0)}VS_F^{(0)} = \int S_F^{(0)}(x,x')V(x'')S_F^{(0)}(x'',x')d^4x''. \tag{42}$$

and in the coincidence limit $x \to x'$

$$S_F^{(1)} = \int \frac{d^4kd^4l}{(2\pi)^6}\gamma^\mu k_\mu e^{-ilx}f_2(l)[(\gamma^0k_0 + \gamma^3k_3) + \frac{1}{2}(\gamma^1(k_1-l_1))$$
$$+ \gamma^2(k_2-l_2)\frac{\gamma^\nu(k-l)_\nu}{k^2(k-l)^2}\delta(l_0)\delta(l_3)] \tag{43}$$

and $< T_{00}^{(1)} >$ is

$$< T_{00}^{(1)} >= -Tr\int \frac{d^4kd^4l}{(2\pi)^6}\gamma^0k_0\gamma^\mu k_\mu e^{-ilx}f_2(l)[(\gamma^0k_0 + \gamma^3k_3)$$
$$+ \frac{1}{2}(\gamma^1(k_1-l_2)) + \gamma^2(k_2-l_2)\frac{\gamma^\nu(k-l)_\nu}{k^2(k-l)^2}\delta(l_0)\delta(l_3)] \tag{44}$$

By using the dimensional regularization scheme we get non-zero result for the vacuum expectation value of the energy-momentum tensor

$$< T_{00}^{(1)}(x) >= -\frac{4}{(2\pi)^6}[(-\frac{1}{2}\overline{f}_2(l) + f_2(l))\frac{\Gamma(\frac{\epsilon}{2}-1)}{\Gamma(4)}$$
$$+ 5f_2(l)\frac{\Gamma(\frac{\epsilon}{2}-2)}{\Gamma(6)}]\int d^2l(l^2)^{2-\frac{\epsilon}{2}}e^{-il} \tag{45}$$

Here we have used the definitions of the functions of $f_2(x)$ and $\overline{f}_2(x)$

$$f_2(x) = \beta\log(1+\alpha((x_1-\gamma^1\gamma^2x_2) - \frac{1}{(x_1-\gamma^1\gamma^2x_2)}) \tag{46}$$

and

$$\overline{f}_2(x) = \beta \log(1 + \alpha((x_1 + \gamma^1\gamma^2 x_2) - \frac{1}{(x_1 + \gamma^1\gamma^2 x_2)}) \tag{47}$$

In equation (45) $f_2(l)$ and $\overline{f}_2(l)$is the Fourier transformation of the functions $f_2(x)$ and $\overline{f}_2(x)$ respectively. Our calculations have been done for the small parameter β and α. The expansion of the logarithm to the first order in α gives us $f_2(x)$ and $\overline{f}_2(x)$:

$$f_2(x) \approx \beta \log(1 + \alpha((x_1 - \gamma^1\gamma^2 x_2) - \frac{1}{(x_1 - \gamma^1\gamma^2 x_2)}) \tag{48}$$

and

$$\overline{f}_2(x) \approx \beta \log(1 + \alpha((x_1 + \gamma^1\gamma^2 x_2) - \frac{1}{(x_1 + \gamma^1\gamma^2 x_2)}) \tag{49}$$

Finally substituting the Fourier transformation of these functions in (45) we get non-zero result for $< T_{00}^{(1)} >$

$$< T_{00}^{(1)}(x) >= -\frac{\beta\alpha}{8\pi} \frac{x_1}{(x_1^2 + x_2^2)^3} \tag{50}$$

This shows the expected behaviour in the stringy space-time.

5 Conclusion

Here we have shown how perturbative work on multiple cosmic strings can be extended to other locally flat metrics and explicitly calculated the dipole case for the scalar and the spinor cases.

The same formalism can be applied to other configurations. A simple extension will be

$$g_2 = -\beta \log\left(1 + \alpha(z^2 - \frac{1}{z^2})\right) \tag{51}$$

which corresponds to six cosmic strings, two equally spaced on the x_1 axis on two sides of the origin, two more equally spaced on two sides of the origin

on the x_2 axis, and two cosmic strings which result in excess angles, located on the origin of the $x_1 - x_2$ plane. Then $< T_{00} >$ will be proportional to

$$\frac{1}{(x^2+y^2)^3}[1-\frac{2x^2}{x^2+y^2}].$$

The same expression will also appear in the α^2 term in the expansion of the dipole case.

By playing with the arbitrary functions one can get more interesting cases, however, it is not so easy to obtain other than Dirac delta function type singularities for the Ricci scalar. Our metrics are just examples of locally flat metrics. This behaviour is also seen in the fact that we can take the "square root" of the d'Alembertian operator in this case. If we use the Euclidean metric this is also possible for $f_1 \neq 0$ when g_1 is choosen in the form given by equation 4. This would allow the study of "non-static" cosmic string-like metrics, if we can interprete "euclidean time" as time.

At this point one may note that solutions with torsion, as well as curvature singularities may be obtained [/8]. It may be interesting to generalize our metrics in this direction.

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